

PARTICLES AS BOUND STATES IN THEIR OWN POTENTIALS[†]**R. P. Woodard**

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ABSTRACT

I consider the problem of computing the mass of a charged, gravitating particle in quantum field theory. It is shown how solving for the first quantized propagator of a particle in the presence of its own potentials reproduces the gauge and general coordinate invariant sum over an infinite class of diagrams. The distinguishing feature of this class of diagrams is that all closed loops contain part of the continuous matter line running from early to late times. The next order term would have one closed loop external to the continuous matter line, and so on. I argue that the gravitational potentials in the 0-th order term may permit the formation of bound states, which would then dominate the propagator. It is conceivable that this provides an tractable technique for computing the masses of fundamental particles from first principles. It is also conceivable that the expansion in external loops permits gravity to regulate certain ultraviolet divergences.

1. INTRODUCTION: A PARABLE OF POLITICAL CORRECTNESS

I wish to speak out against a form of bigotry. The prejudice in question might be termed, *integro-centrism*, and it consists of the belief that asymptotic series may contain only non-negative, integer powers of the expansion coefficient. Not only is this exclusionary against non-integer powers and logarithms, it even discriminates against sign-challenged integers! I shall also argue that integro-centrism may be imposing a kind of cultural genocide on quantum gravity and on the problem of mass.

Imagine that you are the asymptotic expansion $\tilde{f}(g)$ of some quantum field theoretic quantity $f(g)$. Without succumbing to negative stereotypes we can assume you have the following form:

$$\tilde{f}(g) = \sum_{n=0}^{\infty} f_n \phi_n(g). \quad (1.1)$$

We can also assume that the $\phi_n(g)$ are elementary functions which have been arranged in

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a (value-neutral) order such that:

$$\lim_{g \rightarrow 0} \frac{\phi_{n+1}(g)}{\phi_n(g)} = 0 . \quad (1.2)$$

Finally, the fact that you are asymptotic means that the difference between $f(g)$ and the sum of your first N terms must vanish faster than your N -th term as g goes to zero:

$$\lim_{g \rightarrow 0} \left(f(g) - \sum_{n=0}^N f_n \phi_n(g) \right) \frac{1}{\phi_N(g)} = 0 . \quad (1.3)$$

In the Ward and June Cleaver world of conventional perturbation theory the coefficient functions would be integer powers — $\phi_n(g) = g^n$ — and their coefficients could be obtained by taking derivatives of the original function $f(g)$:

$$f_n = \frac{1}{n!} \frac{\partial^n f(g)}{\partial g^n} \Big|_{g=0} . \quad (1.4)$$

Suppose, however, that you are leading an alternate lifestyle which includes logarithms or fractional powers. For example, you might have the form:

$$\tilde{f}(g) = 1 + 3g \ln(g) + O(g^2) . \quad (1.5)$$

Although your actual first order correction is small for small g , an integro-centric bigot would claim it is logarithmically divergent:

$$f_1 = \lim_{g \rightarrow 0} \left(3 \ln(g) + 3 + O(g) \right) . \quad (1.6)$$

And he would compute the higher terms to consist of an oscillating tower of increasingly virulent divergences:

$$f_n = \lim_{g \rightarrow 0} \left(-\frac{3}{n} (-g)^{1-n} + \dots \right) , \quad n \geq 2 . \quad (1.7)$$

His frustration with your non-conformism might provoke him to abandon the quantum field theory behind $f(g)$ in favor of some yet-to-be-specified model in a peculiar dimension. He might even take to making optimistic pronouncements about our ability to exactly solve this model, and hence its correspondence limits of Yang-Mills and General Relativity, in 5-10 years (3-8 years from now, and counting).

Aside from poking fun at the political and scientific prejudices of my colleagues this paper does have some serious points to make. The first of these is that there is no reason why the perturbative non-renormalizability^{1–5} of General Relativity necessarily implies the need for an alternate theory of quantum gravity. It has long been realized that the problem could derive instead from the appearance of logarithms or fractional powers of Newton's constant in the correct asymptotic expansion of quantum gravity.^{6–9} To underscore that this would not be without precedent I devote Section 2 to a discussion of the analogous phenomenon in two simple systems from statistical mechanics.

The second point I wish to make is that the breakdown of conventional perturbation theory in quantum gravity is likely to be associated with ultraviolet divergences. The idea

is that gravity screens effects which tend to make the stress tensor divergent. If so, it must be that the divergence returns when Newton's constant goes to zero, which means the correct asymptotic series must contain logarithms or negative powers. There is no doubt that this does occur on the classical level. Arnowit, Deser and Misner found an explicit example in the finite self-energy of point charged particles.¹⁰ Section 3 is devoted to a brief review of their result.

So far I have been discussing old stuff. Although many people have suspected that quantum gravity regulates ultraviolet divergences^{6–9} no one has been able to make anything of the idea for want of a non-perturbative calculational technique. Divergences *do* evoke an infinite response from gravitation, but only at the next order in perturbation theory. What is needed is a way of reorganizing perturbation theory so that the gravitational response has a chance of keeping up with divergences. The main point of this paper is that I have found such a reorganization, at least for the special case of certain types of matter self-energies.

I begin the derivation in Section 4 by writing down an exact functional integral representation for the mass of a charged, gravitating scalar in quantum field theory. In Section 5 I show that this expression reduces, in the classical limit, to the point particle system studied by ADM,¹⁰ with an extra term representing the negative pressure needed to hold the point charge together. In Section 6 I return to the original, exact expression, and show how it can be rearranged to give an expansion in the number of closed loops which do not include at least some part of the incoming and outgoing matter line. Further, the 0-th order term in this new expansion has the simple interpretation of computing the binding energy of a quantum mechanical particle which moves in the gravitational and electromagnetic potentials induced by its own probability current. This is the origin of the title.

Gravitational attraction must overcome electrostatic repulsion in order for a particle to bind to its own potentials. In Section 7 I obtain the unsurprising result that this can only happen for a scalar which has a Planck scale mass. In the final section I argue that substantially lighter masses may be obtainable for particles with spin.

2. TWO EXAMPLES FROM STATISTICAL MECHANICS

Exotic terms occur in many familiar asymptotic expansions. Consider the logarithm of the grand canonical partition function for non-interacting, non-relativistic bosons in a three dimensional volume V :

$$\ln(\Xi) = V n_Q \sum_{k=1}^{\infty} k^{-\frac{5}{2}} \exp(k\beta\mu). \quad (2.1)$$

Here n_Q is the quantum concentration, μ is the chemical potential, and $\beta = (k_B T)^{-1}$. Near condensation one has $0 < -\beta\mu \ll 1$ so it should make sense to expand $\ln(\Xi)$ for small $\beta\mu$. Straightforward perturbation theory corresponds to the following expansion:

$$\ln(\Xi) = V n_Q \sum_{k=1}^{\infty} k^{-\frac{5}{2}} \sum_{\ell=0}^{\infty} (k\beta\mu)^{\ell} \quad (2.2a)$$

$$\rightarrow V n_Q \sum_{\ell=0}^{\infty} (\beta\mu)^{\ell} \sum_{k=1}^{\infty} k^{\ell-\frac{5}{2}}. \quad (2.2b)$$

Although the $\ell = 0$ and $\ell = 1$ terms are finite, the sum over k diverges for $\ell \geq 2$.

The divergences we have encountered do not mean that higher corrections are large,

just that they are not as small as $(\beta\mu)^2$. One sees this by expanding the second derivative around its integral approximation:

$$\frac{\partial^2 \ln(\Xi)}{\partial(\beta\mu)^2} = V n_Q \sum_{k=1}^{\infty} k^{-\frac{1}{2}} \exp(k\beta\mu) \quad (2.3a)$$

$$= V n_Q \left\{ \int_0^{\infty} dy y^{-\frac{1}{2}} \exp(y\beta\mu) + \sum_{k=1}^{\infty} \left[k^{-\frac{1}{2}} \exp(k\beta\mu) - \int_{k-1}^k dy y^{-\frac{1}{2}} \exp(y\beta\mu) \right] \right\} \quad (2.3b)$$

$$= V n_Q \left\{ \left(\frac{-\pi}{\beta\mu} \right)^{\frac{1}{2}} + \sum_{k=1}^{\infty} \left[k^{-\frac{1}{2}} - 2 k^{\frac{1}{2}} + 2 (k-1)^{\frac{1}{2}} \right] + O(\beta\mu) \right\}. \quad (2.3c)$$

Integration reveals the true asymptotic expansion:

$$\ln(\Xi) = V n_Q \left\{ \zeta\left(\frac{5}{2}\right) + \zeta\left(\frac{3}{2}\right) \beta\mu + \frac{4}{3} \sqrt{\pi} (-\beta\mu)^{\frac{3}{2}} + O(\beta^2\mu^2) \right\}. \quad (2.4)$$

The oscillating series of ever-increasing divergences in the perturbative expansion (2.2b) has resolved itself into a perfectly finite, fractional power.

Logarithms can also invalidate perturbation theory. Consider the canonical partition function for a non-interacting particle of mass m in a three dimensional volume V :

$$Z = \frac{V}{2\pi^2\hbar^3} \int_0^{\infty} dp p^2 \exp\left[-\beta\sqrt{p^2 c^2 + m^2 c^4} + \beta mc^2\right] \quad (2.5a)$$

$$= \frac{V}{2\pi^2\hbar^3 c^3} \int_0^{\infty} dK (K + mc^2) \sqrt{K^2 + 2 K mc^2} \exp(-\beta K). \quad (2.5b)$$

When the rest mass energy is small compared to the thermal energy it ought to make sense to expand in the small parameter $x \equiv \beta mc^2$. But straightforward perturbation theory fails again:

$$Z = \frac{V}{2\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3 \int_0^{\infty} dt t^2 e^{-t} \left(1 + \frac{x}{t} \right) \sqrt{1 + 2\frac{x}{t}} \quad (2.6a)$$

$$= \frac{V}{2\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3 \int_0^{\infty} dt t^2 e^{-t} \left\{ 1 + 2\frac{x}{t} + \frac{1}{2} \left(\frac{x}{t} \right)^2 - \sum_{n=3}^{\infty} \frac{(n-3)(2n-5)!!}{n!} \left(-\frac{x}{t} \right)^n \right\}. \quad (2.6b)$$

It seems as though the term of order x^3 vanishes, and that the higher terms have increasingly divergent coefficients with oscillating signs. In fact the x^3 term is non-zero, and the apparent divergences merely signal contamination with logarithms:

$$Z = \frac{V}{2\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3 \left\{ 2 + 2x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{48}x^4 \ln(x) + O(x^4) \right\}. \quad (2.7)$$

3. THE ADM MECHANISM

Arnowitt, Deser and Misner showed that perturbation theory also breaks down in computing the self-energy of a classical, charged, gravitating point particle.¹⁰ It is simplest to model the particle as a stationary spherical shell of radius ϵ , charge e and bare mass m_0 . In Newtonian gravity its energy would be:

$$E = m_0 + \frac{e^2}{8\pi\epsilon} - \frac{Gm_0^2}{2\epsilon}. \quad (3.1)$$

It turns out that all the effects of general relativity are accounted for by replacing E and m_0 with the full mass:^{*}

$$m_\epsilon = m_0 + \frac{e^2}{8\pi\epsilon} - \frac{Gm_\epsilon^2}{2\epsilon} \quad (3.2a)$$

$$= \frac{\epsilon}{G} \left[-1 + \sqrt{1 + \frac{2G}{\epsilon} \left(m_0 + \frac{e^2}{8\pi\epsilon} \right)} \right]. \quad (3.2b)$$

The perturbative result is obtained by expanding the square root:

$$m_{\text{pert}} = m_0 + \frac{e^2}{8\pi\epsilon} + \sum_{n=2}^{\infty} \frac{(2n-3)!!}{n!} \left(-\frac{G}{\epsilon} \right)^{n-1} \left(m_0 + \frac{e^2}{8\pi\epsilon} \right)^n, \quad (3.3)$$

and shows the oscillating series of increasingly singular terms characteristic of the previous examples. The alternating sign derives from the fact that gravity is attractive. The positive divergence of order e^2/ϵ evokes a negative divergence or order Ge^4/ϵ^3 , which results in a positive divergence of order G^2e^6/ϵ^5 , and so on. The reason these terms are increasingly singular is that the gravitational response to an effect at one order is delayed to a higher order in perturbation theory.

The correct result is obtained by taking ϵ to zero before expanding in the coupling constants e^2 and G :

$$\lim_{\epsilon \rightarrow 0} m_\epsilon = \left(\frac{e^2}{4\pi G} \right)^{\frac{1}{2}}. \quad (3.4)$$

Like the examples of Section 2 it is finite but not analytic in the coupling constants e^2 and G . Unlike the previous examples, it diverges for small G . This is because gravity has regulated the linear self-energy divergence which results for a non-gravitating charged particle.

One can understand the process from the fact that gravity has a built-in tendency to oppose divergences. A charge shell does not want to contract in pure electromagnetism; the act of compressing it calls forth a huge energy density concentrated in the nearby electric field. Gravity, on the other hand, tends to make things collapse, especially large concentrations of energy density. The dynamical signature of this tendency is the large

* It should be noted that Arnowitt, Deser and Misner rigorously solved the constraint equations of general relativity and electrodynamics, and then used the asymptotic metric to compute the ADM mass.¹⁰ They also developed the simple model I am presenting.

negative energy density concentrated in the Newtonian gravitational potential. In the limit the two effects balance and a finite total mass results.

Said this way, there seems no reason why gravitational interactions should not act to cancel divergences in quantum field theory. It is especially significant, in this context, that the divergences of some quantum field theories — such as QED — are weaker than the linear ones which ADM have shown that classical gravity regulates. The frustrating thing is that one cannot hope to see the cancellation perturbatively. In perturbation theory the gravitational response to an effect at any order must be delayed to a higher order. This is why the perturbative result (3.3) consists of an oscillating series of ever higher divergences. What is needed is an approximation technique in which the gravitational response is able to keep pace with what is going on in other sectors.

A final point of interest is that any finite bare mass drops out of the exact result (3.4) in the limit $\epsilon \rightarrow 0$. This makes for an interesting contrast with the usual program of renormalization. Without gravity one would pick the desired physical mass, m_p , and then adjust the bare mass to be whatever divergent quantity was necessary to give it:

$$m_0 = m_p - \frac{e^2}{8\pi\epsilon} . \quad (3.5)$$

Of course the same procedure would work with gravity as well:

$$m_0 = m_p - \frac{e^2}{8\pi\epsilon} + \frac{Gm_p^2}{2\epsilon} . \quad (3.6)$$

The difference with gravity is that we have an alternative: keep m_0 finite and let the dynamical cancellation of divergences produce a unique result for the physical mass. The ADM mechanism is in fact the classical realization of the old dream of computing a particle's mass from its self-interactions.

4. MASS OF A CHARGED GRAVITATING SCALAR IN QFT

The purpose of this section is to obtain a convenient functional integral representation for the standard quantum field theoretic definition of a particle's mass as the pole of its propagator. For simplicity I will consider a charged, gravitating scalar, the Lagrangian for which is:

$$\begin{aligned} \mathcal{L} = \frac{1}{16\pi G} & \left(R\sqrt{-g} - \text{S.T.} \right) - \frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} \\ & - (\partial_\mu - iqA_\mu) \phi^* (\partial_\nu + iqA_\nu) \phi g^{\mu\nu} \sqrt{-g} - m_0^2 \phi^* \phi \sqrt{-g} . \end{aligned} \quad (4.1a)$$

The symbol “S.T.” denotes the gravitational surface term needed to purge the Lagrangian of second derivatives:

$$\text{S.T.} \equiv \partial_\mu \left[(g_{\nu\rho,\sigma} - g_{\rho\sigma,\nu}) g^{\mu\nu} g^{\rho\sigma} \sqrt{-g} \right] . \quad (4.1b)$$

If we temporarily regulate infrared divergences and agree to understand operator relations in the weak sense then it is possible to write the operators which annihilate outgoing particles and create incoming ones as simple limits:^{*}

$$a_k^{\text{out}} = \lim_{t_+ \rightarrow \infty} \frac{ie^{i\omega t_+}}{\sqrt{2\omega Z}} \overleftrightarrow{W}_+ \tilde{\phi}(t_+, \vec{k}) , \quad (4.2a)$$

* The notation employed in these formulae is standard: Z is the field strength renormalization, the Wronskian is $\overleftrightarrow{W} \equiv \overrightarrow{\partial}_0 - \overleftarrow{\partial}_0$, and a tilde over the scalar field denotes its spatial Fourier transform.

$$\left(a_k^{\text{in}}\right)^+ = \lim_{t_- \rightarrow -\infty} \frac{ie^{-i\omega t_-}}{\sqrt{2\omega Z}} \overleftrightarrow{W}_- \tilde{\phi}^*(t_-, \vec{k}) , \quad (4.2b)$$

where the energy is:

$$\omega \equiv \sqrt{k^2 + m^2} . \quad (4.3)$$

Consider single particle states whose wave functions in the infinite past and future are ψ_{\mp} , respectively. The inner product between two such states can be given the following expression:

$$\begin{aligned} \langle \psi_+^{\text{out}} | \psi_-^{\text{in}} \rangle &= \int \frac{d^3 k}{(2\pi)^3} \frac{\psi_+^*(\vec{k}) \psi_-(\vec{k})}{2\omega Z} \lim_{t_{\pm} \rightarrow \pm\infty} e^{i\omega(t_+ - t_-)} \overleftrightarrow{W}_+ \overleftrightarrow{W}_- \\ &\times \int d^3 x e^{-i\vec{k} \cdot \vec{x}} \langle \Omega^{\text{out}} | \phi(t_+, \vec{x}) \phi^*(t_-, \vec{0}) | \Omega^{\text{in}} \rangle . \end{aligned} \quad (4.4)$$

One way of computing the mass is to tune the parameter m in the energy (4.3) to the precise value for which expression (4.4) assumes the form:

$$\langle \psi_+^{\text{out}} | \psi_-^{\text{in}} \rangle = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega} \psi_+^*(\vec{k}) \psi_-(\vec{k}) . \quad (4.5)$$

This agrees with the usual definition of the mass as the pole of the propagator.

A somewhat more direct way of computing the mass is to focus on the second line of (4.4) which we can write as a phase:

$$e^{-i\xi(t_+, t_-, k)} \equiv \int d^3 x e^{-i\vec{k} \cdot \vec{x}} \langle \Omega^{\text{out}} | \phi(t_+, \vec{x}) \phi^*(t_-, \vec{0}) | \Omega^{\text{in}} \rangle . \quad (4.6)$$

Dividing by the time interval and then taking it to infinity we obtain the energy:

$$\lim_{t_{\pm} \rightarrow \pm\infty} \left(\frac{\xi(t_+, t_-, k)}{t_+ - t_-} \right) = \sqrt{k^2 + m^2} . \quad (4.7)$$

Note that by using this method we avoid the problem of infrared divergences. These affect only the field strength renormalization, not the mass.

It is straightforward to write the phase as a functional integral:

$$e^{-i\xi} = \int d^3 x e^{-i\vec{k} \cdot \vec{x}} \left[[dg][dA][d\phi] \phi(t_+, \vec{x}) \phi^*(t_-, \vec{0}) e^{iS_{\text{GR}}[g] + iS_{\text{EM}}[g, A] + iS_{\phi}[g, A, \phi]} \right] . \quad (4.8)$$

The next step is to integrate out the scalar. In the presence of an arbitrary metric and electromagnetic background its kinetic operator is:

$$\mathcal{D}[g, A] \equiv \frac{1}{\sqrt{-g}} \left(\partial_\mu + iqA_\mu \right) \sqrt{-g} g^{\mu\nu} \left(\partial_\nu + iqA_\nu \right) . \quad (4.9)$$

We can use this operator to express the scalar-induced effective action:

$$\Gamma_\phi[g, A] \equiv -i \ln \left(\det[-\mathcal{D} + m_0^2 - i\epsilon] \right) , \quad (4.10a)$$

and the scalar propagator in the presence of an general background:

$$D[g, A](t_+, \vec{x}; t_-, \vec{0}) \equiv \left\langle t_+, \vec{x} \middle| \frac{i}{\mathcal{D} - m_0^2 + i\epsilon} \right| t_-, \vec{0} \rangle . \quad (4.10b)$$

With these objects the phase can be reduced to a functional integral over only metrics and vector potentials:

$$e^{-i\xi} = \int d^3x e^{-i\vec{k}\cdot\vec{x}} \left[[dg][dA] D[g, A](t_+, \vec{x}; t_-, \vec{0}) e^{iS_{\text{GR}}[g] + iS_{\text{EM}}[g, A] + i\Gamma_\phi[g, A]} \right] , \quad (4.11)$$

Contact is made with particle dynamics by writing the general propagator in Schwinger form:

$$\left\langle t_+, \vec{x} \middle| \frac{i}{\mathcal{D} - m_0^2 + i\epsilon} \right| t_-, \vec{0} \rangle = \int_0^\infty ds \left\langle t_+, \vec{x} \middle| \exp \left[is(\mathcal{D} - m_0^2 + i\epsilon) \right] \right| t_-, \vec{0} \rangle . \quad (4.12)$$

One then regards the exponent as the Hamiltonian of a first quantized particle and the expectation value is converted into a functional integral in the usual way. We can give this a reparameterization invariant form by regarding the proper time as the unfixed part of the einbein $e(\tau)$ in $\dot{e} = 0$ gauge:

$$\int_0^\infty ds = \left[[de] \delta[\dot{e}] \right] \quad (4.13)$$

Integrating out the canonical momenta and absorbing any ordering terms into the measure gives:

$$\begin{aligned} & \int d^3x e^{-i\vec{k}\cdot\vec{x}} \left\langle t_+, \vec{x} \middle| \frac{i}{\mathcal{D} - m_0^2 + i\epsilon} \right| t_-, \vec{0} \rangle \\ &= \left[[de][d^4\chi] \delta[\dot{e}] \delta(\chi^0(0) - t_-) \delta(\chi^0(1) - t_+) \delta^3(\vec{\chi}(0)) \right. \\ & \quad \times \exp \left\{ -i\vec{k} \cdot \vec{\chi}(1) + i \int_0^1 d\tau \left[\frac{1}{4e} g_{\mu\nu} \dot{\chi}^\mu \dot{\chi}^\nu - em_0^2 - q\dot{\chi}^\mu A_\mu \right] \right\} \end{aligned} \quad (4.14)$$

One now makes the change of variables defined by the reparameterization which changes the gauge condition to:

$$\chi^0(\tau) \equiv t_- + (t_+ - t_-)\tau . \quad (4.15)$$

The integral over the einbein is done using the functional equivalent of the identity:

$$\int_0^\infty \frac{dx}{\sqrt{x}} \exp \left[-ax - \frac{b}{x} \right] = \sqrt{\frac{\pi}{a}} \exp \left[-\sqrt{4ab} \right] . \quad (4.16)$$

The final form is:

$$\begin{aligned} & \int d^3x e^{-i\vec{k}\cdot\vec{x}} \left\langle t_+, \vec{x} \middle| \frac{i}{\mathcal{D} - m_0^2 + i\epsilon} \right| t_-, \vec{0} \rangle \\ &= \left[[d^3\chi] \delta^3(\vec{\chi}(0)) \exp \left\{ -i\vec{k} \cdot \vec{\chi}(1) - i \int_0^1 d\tau \left[m_0 \sqrt{-g_{\mu\nu} \dot{\chi}^\mu \dot{\chi}^\nu} + q\dot{\chi}^\mu A_\mu \right] \right\} \right] \end{aligned} \quad (4.17)$$

where $\chi^0(\tau)$ is understood to be defined by (4.15).

Substituting (4.17) into (4.11) gives the following expression for the phase:

$$e^{-i\xi} = \int [dg][dA][d^3\chi] \delta^3(\vec{\chi}(0)) \exp\left\{-i\vec{k} \cdot \vec{\chi}(1)\right\} \times \exp\left\{iS_{\text{GR}}[g] + iS_{\text{EM}}[g, A] + iS_{\text{part}}[g, A, \chi] + i\Gamma_\phi[g, A]\right\}, \quad (4.18a)$$

where the particle action is:

$$S_{\text{part}}[g, A, \chi] \equiv - \int_0^1 d\tau \left[m_0 \sqrt{-g_{\mu\nu}(\chi(\tau)) \dot{\chi}^\mu(\tau) \dot{\chi}^\nu(\tau)} + q \dot{\chi}^\mu(\tau) A_\mu(\chi(\tau)) \right]. \quad (4.18b)$$

It should be noted again that various ordering corrections have been subsumed into the measure. Also note, again, that $\chi^0(\tau)$ is the non-dynamical function (4.15).

5. THE CLASSICAL LIMIT

The purpose of this section is to show how the classical limit of ξ relates to the ADM¹⁰ mechanism discussed in Section 3. This is crucial to seeing that the reorganization of perturbation theory I shall propose in the next section in fact manifests the gravitational regulation of ultraviolet divergences at lowest order.

We can forget about the scalar-induced effective action Γ_ϕ because it is a quantum effect. What is necessary for our purposes is to solve the classical field equations derived from the action:^{*}

$$S_{\text{class}}[g, A, \chi] = S_{\text{GR}}[g] + S_{\text{EM}}[g, A] + S_{\text{part}}[g, A, \chi]. \quad (5.1)$$

The boundary conditions for the metric and the vector potential come from the asymptotic in and out vacua. Those for the particle are:

$$\chi^i(0) = 0 \quad , \quad \dot{\chi}^i(1) = k^i. \quad (5.2a)$$

We can save ourselves a small amount of effort by instead imposing:

$$\chi^i(0) = 0 \quad , \quad \dot{\chi}^i(1) = 0, \quad (5.2b)$$

and then boosting up to (5.2a). If (5.2b) is used one finds the classical limit of the phase by evaluating the action at the solution:

$$\xi_{\text{class}}(t_+, t_-, 0) = -S_{\text{class}}[\hat{g}, \hat{A}, \hat{\chi}]. \quad (5.3a)$$

One then divides out the time interval and takes the asymptotic limit:

$$m_{\text{class}} = \lim_{t_\pm \rightarrow \pm\infty} \left(\frac{\xi(t_+, t_-, 0)}{t_+ - t_-} \right). \quad (5.3b)$$

* The same technique has been used, in the context of 2-body scattering, by Fabbrichesi, Pettorino, Veneziano and Vilkovisky¹¹.

Although gravity does regulate this problem, just as it did for that of ADM,¹⁰ some of the intermediate expressions will be singular unless the point particle is smeared out. ADM resolved this issue by converting the particle into a spherical shell of radius ϵ in isotropic coordinates. I shall do the same, but I face the additional problem, which they did not, of keeping the system static for all time. I shall accordingly employ a perfect fluid regularization in which the point particle is converted into a swarm of particles labelled by an internal vector $\vec{\sigma}$:

$$\chi^i(\tau) \longrightarrow X^i(\tau, \vec{\sigma}) \quad (5.4)$$

The particle's action goes to that of a perfect fluid:

$$S_{\text{part}}[g, A, \chi] \longrightarrow - \int d\tau d^3\sigma \left\{ \sqrt{-g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta} \left[\mu(\vec{\sigma}) + \frac{\Pi(\vec{\sigma})}{\sqrt{-g}} \right] + \frac{q}{m_0} \mu(\vec{\sigma}) \dot{X}^\mu A_\mu \right\}, \quad (5.5)$$

with number density $n(x)$ given by $\mu(\vec{\sigma})$:

$$n(x) = \frac{1}{\sqrt{-g}} \int d\tau d^3\sigma \frac{\mu(\vec{\sigma})}{m_0} \delta^4(x - X(\tau, \vec{\sigma})) \sqrt{-g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta}, \quad (5.6a)$$

and pressure $p(x)$ given by $\Pi(\vec{\sigma})$:

$$p(x) = -\frac{1}{g} \int d\tau d^3\sigma \Pi(\vec{\sigma}) \delta^4(x - X(\tau, \vec{\sigma})) \sqrt{-g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta}, \quad (5.6b)$$

I shall follow ADM in taking the mass density to be that of a spherical shell:

$$\mu(\vec{\sigma}) \equiv \frac{m_0 \delta(\sigma - \epsilon)}{4\pi \epsilon^2}, \quad (5.7a)$$

however, I shall impose a negative internal pressure:

$$\Pi(\vec{\sigma}) \equiv -f(\epsilon) \theta(\epsilon - \sigma), \quad (5.7b)$$

to hold the shell together. Duff has shown that this does not affect the ADM mass.¹² The function $f(\epsilon)$ is a non-dynamical constant to be determined shortly.

The manifest spherical symmetry and the assumed time translation invariance of this problem suggest that we look for a solution of the form:

$$\hat{g}_{\mu\nu} dx^\mu dx^\nu = -A^2(r) dt^2 + B^2(r) d\vec{x} \cdot d\vec{x}, \quad (5.8a)$$

$$\hat{A}_\mu dx^\mu = A_0(r) dt, \quad (5.8b)$$

$$\hat{X}^\mu(\tau, \vec{\sigma}) = \delta_0^\mu \left(t_- + (t_+ - t_-)\tau \right) + \delta_i^\mu \sigma^i. \quad (5.8c)$$

The solution for $A(r)$ has the form $A(r) = \alpha(r)/B(r)$ with:

$$\alpha(r) = \theta(r - \epsilon) \left[1 + \frac{Q^2 - M^2}{4r^2} \right] + \theta(\epsilon - r) \left[1 + \left(\frac{Q^2 - M^2}{4\epsilon^2} \right) \left(2 - \frac{r^2}{\epsilon^2} \right) \right]. \quad (5.9a)$$

The two other functions work out to be:

$$B(r) = \theta(r - \epsilon) \left[1 + \frac{M}{r} - \left(\frac{Q^2 - M^2}{4r^2} \right) \right] + \theta(\epsilon - r) \left[1 + \frac{M}{\epsilon} - \left(\frac{Q^2 - M^2}{4\epsilon^2} \right) \right], \quad (5.9b)$$

$$A_0(r) = \frac{q}{4\pi B(r)} \left\{ \frac{\theta(r-\epsilon)}{r} + \frac{\theta(\epsilon-r)}{\epsilon} \right\}. \quad (5.9c)$$

The parameters in relations (5.9a-c) have been represented as lengths according to the standard convention of geometrodynamics:

$$M_0 \equiv Gm_0 \quad , \quad Q \equiv q\sqrt{\frac{G}{4\pi}}, \quad (5.10a)$$

$$M \equiv \epsilon \left[-1 + \sqrt{1 + \frac{2M_0}{\epsilon} + \frac{Q^2}{\epsilon^2}} \right]. \quad (5.10b)$$

The necessary internal pressure constant $f(\epsilon)$ works out to be:

$$f(\epsilon) = \frac{B^3(\epsilon)}{8\pi G} \left(\frac{Q^2 - M^2}{\epsilon^4} \right). \quad (5.11)$$

Since the solution is static the action must be a constant multiplied by the time interval. However, this constant turns out to be minus the ADM mass but rather a sort of enthalpy reflecting the presence of the pressure in the perfect fluid regularization of the particle action:

$$\left(S_{\text{GR}} + S_{\text{EM}} + S_{\text{part}} \right) \Big|_{\text{solution}} = -\frac{M}{G}(t_+ - t_-) - \int d^4x \sqrt{-g} p, \quad (5.12a)$$

$$\equiv -(U + pV)(t_+ - t_-). \quad (5.12b)$$

The energy U and the pV term can be evaluated for any ϵ . They have the following simple forms:

$$U = \frac{M}{G} \quad , \quad pV = -\frac{1}{3G}(M - M_0). \quad (5.13)$$

In the limit $\epsilon \rightarrow 0$ the energy just gives the ADM mass (3.4). The pV term remains finite in this limit, but neither does it vanish. Its physical interpretation seems to be that gravity is not sufficient to hold the charge together. This means the calculation is not really consistent. I shall do better shortly, but do not let this obscure two important facts:

- (1) Contact has been established, modulo the pV term, between the standard quantum field theoretic definition of mass and the classical ADM calculation of the self-energy of a gravitating, point charged particle.
- (2) Even with the pV term, gravity has suppressed what would otherwise be a divergent result.

6. QUANTUM MECHANICAL INTERPRETATION

The purpose of this section is to introduce the promised reorganization of conventional perturbation theory. The starting point is the expression (4.18a) obtained for the phase at the end of Section 4:

$$\begin{aligned} e^{-i\xi} &= \left[[dg][dA] \exp \left\{ iS_{\text{GR}}[g] + iS_{\text{EM}}[g, A] + i\Gamma_\phi[g, A] \right\} \right. \\ &\quad \times \left. [d\chi] \delta^3(\vec{\chi}(0)) \exp \left\{ -i\vec{k} \cdot \vec{\chi}(1) + iS_{\text{part}}[g, A, \chi] \right\} \right]. \end{aligned} \quad (6.1)$$

The second line of this expression can be interpreted as the amplitude for a quantum mechanical particle to go from a delta function at $t = t_-$:

$$\psi_-(\vec{x}) = \delta^3(\vec{x}), \quad (6.2a)$$

to a plane wave at $t = t_+$:

$$\psi_+(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}, \quad (6.2b)$$

Let us denote the associated action as:

$$\exp\left\{iS_{\text{prop}}[g, A]\right\} \equiv \int [d\chi] \delta^3(\vec{\chi}(0)) \exp\left\{-i\vec{k}\cdot\vec{\chi}(1) + iS_{\text{part}}[g, A, \chi]\right\}. \quad (6.3)$$

I define the 0-th order term in the reorganized perturbation theory to be the stationary phase approximation to the functional integral over metrics and vector potentials with the following action:

$$S_{\text{class}}[g, A] = S_{\text{GR}}[g] + S_{\text{EM}}[g, A] + S_{\text{prop}}[g, A]. \quad (6.4)$$

To see what diagrams this term captures it is simplest to identify the ones it misses. No closed scalar loops are included since the scalar-induced effective action Γ_ϕ was excluded from (6.4). This does not mean there are no scalar lines at all. Owing to the presence of S_{prop} there must be a single, continuous scalar line in all the included diagrams. Any number of graviton and gauge lines can be attached to this line. However, the restriction to stationary phase means that we include no closed gauge and/or graviton loops which do not include some portion of the single, continuous scalar line. *So the 0-th order term I have defined consists of all diagrams with a single, continuous scalar line and no closed loops which do not include this line.* One can imagine that the next order term consists of diagrams with one closed loop external to the in-out line, and so on.

It is obvious from the way I have defined it that the 0-th order term constitutes a gauge invariant resummation of an infinite subset of diagrams. It is also obvious this 0-th term contains the classical limit considered in the previous section. Further, it will be rendered *less* singular, not more, by the inevitable quantum spread of the particle. All of this implies that the 0-th term just defined must manifest the gravitational suppression of divergences.

The physical interpretation of the 0-th order approximation to ξ is the phase developed by a quantum mechanical particle moving in the potentials generated by its own probability current. Whether or not there is any chance of being able to compute it depends upon which of the following two possibilities is realized:

- (1) The particle cannot form bound states in its own potentials; or
- (2) The particle can form bound states in its own potentials.

In case (1) we are left with a complicated, time dependent scattering problem which seems to be intractable. However, many simplifications are possible in case (2).

If bound states form one can forget about the asymptotic wavefunctions (6.2a-b). In the limit of infinite time separation the phase will be dominated by the lowest energy state. Further, one need only compute $S_{\text{prop}}[g, A]$ for a class of metrics and vector potentials which is broad enough to include the eventual solution. In the scalar problem we could immediately reduce from nine functions of x^μ to a static, spherically symmetric system characterized by only three functions of a single variable:

$$g_{\mu\nu}dx^\mu dx^\nu = -A^2(r)dt^2 + B^2(r)d\vec{x} \cdot d\vec{x} \quad (6.5a)$$

$$A_\mu dx^\mu = A_0(r)dt \quad (6.5b)$$

Finally, variational methods can be usefully applied. If one simply guesses the wavefunction, assuming static potentials, and then minimizes the total energy, the result will be an upper bound on the true 0-th order mass. Note, in this context, that *any* finite result would be awe inspiring.

7. QUANTUM MECHANICS IN REISSNER-NORDSTROM

The purpose of this section is to ascertain which of the two cases pertains to a charged, gravitating scalar: can it bind to its own potentials or not? We can immediately specialize to the static, spherically symmetric potentials (6.5a-b). Modulo the effects of operator ordering, the Hamiltonian is:

$$H = \frac{A(r)}{B(r)} \sqrt{\|\vec{p}\|^2 + m_0^2 B^2(r)} + qA_0(r) \quad (7.1)$$

Assuming the particle is bound, we can invoke Birkhoff's theorem to fix the potentials outside most of the particle's probability density:

$$A_{\text{ext}}(r) = \frac{1}{B_{\text{ext}}} \left[1 + \left(\frac{Q^2 - M^2}{4r^2} \right) \right], \quad (7.2a)$$

$$B_{\text{ext}}(r) = 1 + \frac{M}{r} - \left(\frac{Q^2 - M^2}{4r^2} \right), \quad (7.2b)$$

$$A_0^{\text{ext}}(r) = \frac{q}{4\pi r B_{\text{ext}}}. \quad (7.2c)$$

The various parameters have been expressed as lengths in the usual geometrodynamical convention:

$$Q \equiv q\sqrt{\frac{G}{4\pi}} \quad , \quad M_0 \equiv m_0 G, \quad (7.3)$$

however, it should be noted that M is at this stage undetermined. We can also assume that the momentum is dominated by uncertainty pressure:

$$p \sim \frac{\hbar}{r} \quad \implies P \equiv G p \sim \frac{L_P^2}{r}, \quad (7.4)$$

where L_P is the Planck length.

If we geometrodynamicalize the Hamiltonian ($\mathcal{H} \equiv GH$), then its form beyond most of the probability density is:

$$\mathcal{H}_{\text{ext}} = \frac{A_{\text{ext}}}{B_{\text{ext}}} \sqrt{P^2 + M_0^2 B_{\text{ext}}^2} + \frac{Q^2}{r B_{\text{ext}}} \quad (7.5)$$

At large r this becomes:

$$\mathcal{H}_{\text{ext}} \sim M_0 + (Q^2 - M_0 M) \frac{1}{r} \quad r \gg Q \quad (7.6)$$

One consequence of (7.3) is that the ratio Q/L_P goes like the square root of the fine structure constant, whereas the M_0/L_P is the ratio of m_0 to the Planck mass. It follows that any particle which is relevant to low energy physics must obey:

$$M_0, M \ll Q \quad (7.7)$$

In this case we see that the Hamiltonian falls off asymptotically, suggesting that no reasonably light bound state can form.

It is not really consistent to use the external potentials in the interior but doing so fails to reveal an inner region of binding. In isotropic coordinates the singularity occurs at $r_0 = (Q - M)/2$. Specializing to a point just slightly outside gives only another repulsive Hamiltonian:

$$\mathcal{H}_{\text{ext}} \sim \frac{(Q - M)^2 L_P^2}{4Q\epsilon^2} \quad r = \frac{1}{2}(Q - M) + \epsilon \quad (7.8)$$

It seems fair to conclude that any charged scalar bound states would necessarily have Planck scale masses. On the other hand, setting $Q = 0$ in (7.6) seems to suggest that the chargeless scalar can form a bound state. Expression (7.8) suggests that quantum uncertainty pressure protects it from collapse, unlike the neutral scalar studied classically by ADM.¹⁰

8. DISCUSSION

I have proposed a gauge invariant reorganization of conventional perturbation theory in which gravitational regularization is a 0-th order effect. The existence of any new technique deserves comment because it might be thought that the possibilities for one have been pretty well exhausted by now. A new expansion must be in terms of some parameter, such as the dimension of spacetime¹³ or the number of matter fields,¹⁴ and there simply aren't any plausible parameters that have not been tried.

The secret of my expansion is that it does not conform to the usual rules which require the parameter to appear in the Lagrangian. I have instead exploited a parameter which depends, to some extent, on the thing being computed. This parameter is the number of closed loops which are external to continuous matter lines that come in from the asymptotic past and proceed out to the asymptotic future. Not all processes have such lines. However, the technique can be used on those that do, and any evidence for the non-perturbative viability of quantum General Relativity would be interesting.

Of particular interest are the self-energies of matter particles. The classical computation of ADM,¹⁰ summarized in Section 3, suggests that this is a natural setting for conventional perturbation theory to break down. If quantum gravity regulates ultraviolet divergences as classical gravity certainly does then the asymptotic expansion must contain inverse powers or logarithms.

I was able to reexpress the standard definition of the pole of the propagator in terms of the new expansion. The 0-th order term has the physical interpretation of the phase developed by a quantum mechanical particle moving in the potentials induced by its own probability current. If these potentials cannot form bound states one has an intractable scattering problem. However, the 0-th term is eminently calculable if there are bound states. In this case the lowest energy state dominates. One can also assume that the potentials are static, and that they possess simplifying symmetries. If nothing else works, it is always possible to obtain an upper bound on the mass through variational techniques.

The explicit analysis of Section 7 indicates that there is probably not a charged bound state scalar of less than about the Planck mass. However, it does seem possible that light

neutral scalars can form. Adding spin complicates the gravitational and electrodynamic potentials enormously. It also adds a new parameter in the form of the spin-to-mass ratio a . It may be very significant that, whereas the charge parameter completely dominates the mass, the spin-to-mass ratio is larger by almost the same ratio. For an electron one finds:

$$M \sim 10^{-55} \text{ cm} \ll -Q \sim 10^{-34} \text{ cm} \ll a \sim 10^{-11} \text{ cm}, \quad (8.1)$$

so it is not unreasonable to expect large spin-dependent forces. The physical interpretation for this is that different portions of a rapidly spinning body see one another through enormous relative boosts. What is a minuscule matter density in our frame can therefore seem overwhelming from the instantaneous rest frame of a spinning observer.* So there is some hope for getting light fermionic bound states.

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